arXiv:hep-th/9411082v1 11 Nov 1994

IASSNS-HEP-94/93

W_∞ ALGEBRAS AND INCOMPRESSIBILITY IN THE QUANTUM HALL EFFECT *

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ABSTRACT

We discuss how a large class of incompressible quantum Hall states can be characterized as highest weight states of different representations of the W_∞ algebra. Second quantized expressions of the W_∞ generators are explicitly derived in the cases of multilayer Hall states, the states proposed by Jain to explain the hierarchical filling fractions and the ones related by particle-hole conjugation.

The study of planar, charged nonrelativistic fermions in a strong magnetic field finds important applications in condensed matter problems, such as the quantum Hall effect (QHE)[1,2]. Such systems have a further, less obvious connection to (1+1)-dimensional problems, such as the $c=1$ string model[3,4]. In this talk I will outline the emergence of an infinite dimensional algebraic structure, the W_∞ algebra, and its role in the integer and fractional QHE (IQHE and FQHE), based mainly on work presented in refs. 6 and 7.

1. W_∞ algebras for IQHE

As is well known, the spectrum of planar, nonrelativistic fermions in the presence of a transverse, uniform magnetic field B, consists of infinitely degenerate levels, the so-called Landau levels. The energy gap between adjacent Landau levels is $\omega = B/M$, where M is the fermionic mass ($\hbar = c = e = 1$). For large B we can consider the fermions restricted to the lowest Landau level (LLL). We further assume that B is sufficiently strong to align all electronic spins, so we can neglect the spin degree of freedom. In the symmetric gauge $\vec{A}(\vec{x}) = \frac{B}{2}(y,-x)$, the LLL condition can be written as

$$(\partial_z + \tfrac{1}{2}\bar{z})\ \Psi(\vec{x}) = 0 \tag{1.1}$$

where $z = \sqrt{\frac{B}{2}}(x+iy),\ \bar{z} = \sqrt{\frac{B}{2}}(x-iy)$. The LLL wavefunctions are of the form

$$\Psi(\vec{x}) = f(\bar{z}) \exp\left(-\tfrac{1}{2}|z|^2\right) \tag{1.2}$$

* Talk presented in Mt. Sorak Summer School, S. Korea, June 27-July 2, 1994 and XXth Int. Colloquium on Group Theoretical Methods in Physics, Osaka, Japan, July 4- July 9, 1994, to appear in proceedings.

† This work was supported in part by the DOE grants DE-FG02-85ER40231 and DE-FG02-90ER40542

where $f(\bar{z})$ is a polynomial in $\bar{z}$. Upto an exponential factor which can be absorbed in the definition of the measure, the LLL wavefunctions depend only on the antiholomorphic variables $\bar{z}$, while the holomorphic ones become essentially the canonical momenta ($z \to \partial_{\bar{z}}$ after taking into account the appropriate ordering)[8]. This reflects the fact that the original coordinate space of electrons constrained in the LLL becomes the phase space of a one-dimensional system[3].

In a second quantized language the LLL condition can be promoted to an operator equation and the corresponding LLL fermion operator has the form

$$\Psi(\vec{x},t) = \sqrt{\frac{B}{2\pi}} e^{-\frac{1}{2}|z|^2} \sum_{l=0}^{\infty} C_l(t) \frac{\bar{z}^l}{\sqrt{l!}} \equiv \sqrt{\frac{B}{2\pi}} e^{-\frac{1}{2}|z|^2} \psi(\bar{z},t) \tag{1.3}$$

where C_l's are operators which annihilate fermions of angular momentum l and satisfy the usual anticommutation relations $\{C_l^\dagger, C_{l'}\} = \delta_{l,l'}$.

In the absence of an external potential and interactions there is an infinite degeneracy with respect to angular momentum, so the system is symmetric under independent unitary transformations in the space of C's:

$$C_l(t) = u_{lk} C_k(t) = \langle l|u|k\rangle C_k(t) \tag{1.4}$$

The corresponding infinitesimal transformation for the LLL fermion operator is

$$\delta\Psi^I(\vec{x},t) = i\ddagger\xi(\partial_{\bar{z}} + \frac{z}{2}, \bar{z})\ddagger\ \Psi(\vec{x},t) \tag{1.5}$$

where $\xi(z,\bar{z})$ is a real function and $\ddagger\ \ \ddagger$ indicates that the operators $\partial_{\bar{z}} + \frac{z}{2}$ act from the left. These transformations preserve the LLL condition and the particle number, i.e., $\int d\vec{x}\delta\rho(\vec{x},t) = 0$, where $\rho(\vec{x},t) = \Psi^\dagger(\vec{x},t)\Psi(\vec{x},t)$ is the LLL fermion density. The corresponding generators are given by[4–7]

$$\rho[\xi] \equiv \int d^2z e^{-|z|^2} \psi^\dagger(z)\ \ddagger\xi(\partial_{\bar{z}}, \bar{z})\ddagger\ \psi(\bar{z}) \tag{1.6}$$

where $d^2z \equiv \frac{B}{2\pi} dxdy$, and they satisfy an infinite dimensional algebra given by

$$\begin{aligned} [\,\rho[\,\xi_1\,], \rho[\,\xi_2\,]\,] &= \frac{i}{B}\rho[\{\!\{\xi_1,\xi_2\}\!\}] \\ \{\!\{\xi_1,\xi_2\}\!\} &= iB\sum_{n=1}^{\infty} \frac{(-)^n}{n!} \left(\partial_z^n\xi_1 \partial_{\bar{z}}^n\xi_2 - \partial_{\bar{z}}^n\xi_1\partial_z^n\xi_2\right) \end{aligned} \tag{1.7}$$

$\{\!\{\}\!\}$ is the so-called Moyal bracket. This is the W_∞ algebra[9].

In the absence of an external potential and interactions, the W_∞ algebra corresponds to a symmetry of the problem. In realistic situations the electrons are confined in a finite region. In the infinite plane geometry this can be achieved by introducing an external confining potential $V(\vec{x})$, for example a central harmonic oscillator potential. Such a term spoils the infinite degeneracy with respect to angular momentum by assigning higher energy to higher angular momentum states, but the resulting Hamiltonian, $H = \int d^2x V(\vec{x})\rho(\vec{x},t)$, is a member of the W_∞ algebra[4,5].

In this case the W_∞ algebra does not correspond to a symmetry anymore; instead it provides a spectrum generating algebra. In order to illuminate this role of the W_∞ algebra we consider the action of the W_∞ generators on the many-body ground state of electrons filling up the first Landau level, i.e. $\nu = 1$ (where ν is the filling fraction, the ratio between the number of electrons and the degeneracy of the Landau level). The $\nu = 1$ ground state is

$$|\Psi_{\nu=1}\rangle_0 = C_0^\dagger ... C_{N-1}^\dagger |0\rangle \tag{1.8}$$

for N electrons. This forms an incompressible, circular droplet of radius $\sim \sqrt{N/B}$ and uniform density $\rho = B/2\pi$. Compression corresponds to lowering the angular momentum, but since all available states in the LLL are occupied ($\nu = 1$), that would require an electron to jump to a higher Landau level. However, for a large magnetic field, this is not energetically allowed due to the big energy gap. On the other hand, deformations that would result in transitions to states with higher angular momentum are allowed and cost some energy due to the confining potential. These excitations can be generated by the action of W_∞ generators on the ground state. Inspection of the W_∞ generators in the basis $\xi(z,\bar{z}) = z^l \bar{z}^k$ shows that the operators ρ_{lk} decrease the angular momentum for $l > k$ and increase the angular momentum for $l < k$, where $\rho_{lk} \equiv \int d^2 z e^{-|z|^2} \psi^\dagger(z)(\partial_{\bar{z}})^l (\bar{z})^k \psi(\bar{z})$. Thus we find that[5–7]

$$\begin{aligned} \rho_{lk}|\Psi_{\nu=1}\rangle_0 &= 0 \qquad\qquad \text{if} \quad l > k \\ \rho_{lk}|\Psi_{\nu=1}\rangle_0 &= |\Psi > \qquad \text{if} \quad l \leq k \end{aligned} \tag{1.9}$$

where $|\Psi\rangle$ corresponds to excitations of higher angular momentum.

The first line in Eq. (1.9) provides an algebraic statement for the incompressibility of the ground state, by characterizing the ground state as the highest weight state of the W_∞ algebra[5]. In fact we shall show that even for more general filling fractions the incompressibility of the corresponding ground states can be algebraically expressed as the ground state being the highest weight state of a W_∞ algebra.

Before generalizing to other filling fractions I would like to briefly mention the relation between the W_∞ algebra and the algebra of area preserving diffeomorphisms. This can be easily understood in the case of LLL fermions. W_∞ transformations were earlier introduced as unitary transformations. Their classical analogue, therefore, are canonical transformations which preserve the area element of the phase space. Given that the LLL phase space corresponds to the original two-dimensional coordinate space of the system[3], as mentioned earlier, these canonical transformations are the area preserving diffeomorphisms. In terms of excitations one can understand the relation between the two algebraic structures by considering the edge excitations[10], i.e., $k - l \sim O(1)$. These low energy excitations correspond to boundary fluctuations of the ground state droplet and can be described by one-dimensional chiral boson (fermion) fields. In terms of these, and upon restriction to the edge excitations, one can show that the original W_∞ algebra reduces to the algebra of area-preserving diffeomorphisms[4,11].

The previous analysis can be easily extended to the case where the fermions fill up the first n Landau levels, $\nu = n$. A simple analysis of the Hamiltonian and the corresponding single-body energy and angular momentum wavefunctions shows that the fermion operator can be now expanded as

$$\Psi(\vec{x},t) = \sqrt{\frac{B}{2\pi}} e^{-\frac{1}{2}|z|^2} \sum_{I=0}^{n} \sum_{l=0}^{\infty} i^I C_l^I(t) \frac{(z-\partial_{\bar{z}})^I}{\sqrt{I!}} \frac{\bar{z}^l}{\sqrt{l!}} \equiv \sqrt{\frac{B}{2\pi}} e^{-\frac{1}{2}|z|^2} \sum_{I=0}^{n} \psi^I(z,\bar{z},t) \tag{1.10}$$

where I indicates the Landau level and is related to the energy and $l-I$ measures the angular momentum. The operators C_l^I satisfy $\{C_l^{\dagger I}, C_{l'}^{I'}\} = \delta_{I,I'}\delta_{l,l'}$. There are now n mutually commuting W_∞ generators corresponding to independent unitary transformations acting at each Landau level. They are of the form

$$\rho^I[\xi] \equiv \int d^2 z e^{-|z|^2} \psi^{I\dagger}(z,\bar{z}) \ \ddagger \xi(\partial_{\bar{z}}, \bar{z}-\partial_z) \ddagger \ \psi^I(z,\bar{z}) \qquad I = 0,1,..,n \tag{1.11}$$

Their action on the $\nu = n$ ground state $|\Psi_{\nu=n}\rangle_0$, where (for $N' = nN$ electrons)

$$|\Psi_{\nu=n}\rangle_0 = \prod_{I=0}^{n-1} (C_0^{\dagger I} ... C_{N-1}^{\dagger I})|0\rangle, \tag{1.12}$$

is quite similar to Eq. (1.9). We find that[5,7]

$$\begin{aligned} &\rho_{lk}^I |\Psi_{\nu=n}^I\rangle_0 = 0 && \text{if} \quad l > k \qquad I = 0,1,...,n-1 \\ &\rho_{lk}^I |\Psi_{\nu=n}^I\rangle_0 = |\Psi_I > && \text{if} \quad l \le k \qquad I = 0,1,...,n-1 \end{aligned} \tag{1.13}$$

where $\rho_{lk}^I \equiv \int d^2 z e^{-|z|^2} \psi^{I\dagger}(z,\bar{z}) (\partial_{\bar{z}})^l (\bar{z}-\partial_z)^k \psi(z,\bar{z})$ and $|\Psi_I\rangle$ corresponds to excitations of higher angular momentum at the I-th level. As in the $\nu = 1$ case, the incompressibility of the $\nu = n$ ground state is algebraically expressed as the ground state being the highest weight state of a W_∞ algebra. In the next section we shall seek the generalization of this to the fractional quantum Hall states.

2. W_∞ algebras for FQHE

The main experimental feature of both the IQHE and FQHE, namely the appearance of a series of plateaux where the Hall conductivity is quantized and proportional to the filling fraction ν, while the longitudinal conductivity vanishes, is attributed to the existence of a gap, which gives rise to an incompressible ground state. For IQHE, the essential physics can be well understood in terms of noninteracting fermions. The energy gap is the cyclotron energy separating adjacent Landau levels and for a large magnetic field the Coulomb interaction can be neglected. The noninteracting picture is nonapplicable in the case of the FQHE, where the Coulomb interaction among electrons is important in producing an energy gap. Much of our understanding of the FQHE relies on successful trial wavefunctions, such as the Laughlin wavefunctions[12,13] and the ones proposed more recently by Jain[14]. In both cases they correspond to incompressible configurations of uniform density $\rho = \nu B/2\pi$.

Earlier, in the case of the IQHE, we have seen that the incompressibility of the ground state is closely related to the existence of the W_∞ algebra structure. This relation can be extended to the FQHE[6,7,14,15]. I shall first describe the derivation of W_∞ algebras for $\nu = 1/m$ Laughlin states and their relation to $\nu = 1$ W_∞ algebras which will be crucial in constructing similar algebraic structures for quantum Hall fluids of general filling fraction. Here we consider the electrons confined in the lowest Landau level.

The main point in this derivation is the simple observation that the $\nu = 1/m$ Laughlin ground state wavefunction is related to the $\nu = 1$ wavefunction by attaching $2p$ (where $m = 2p + 1$) flux quanta to each electron

$$\Psi^0_{\nu=1/m} = \prod_{i<j} (\bar{z}_i - \bar{z}_j)^{2p} \Psi^0_{\nu=1} \tag{2.1}$$

Based on this we find that the W_∞ generators for $\nu = 1/m$ are related to the $\nu = 1$ generators by a similarity transformation. The corresponding second quantized expression can be written in terms of fermion and quasihole operators as[6]

$$W_{2p}[\xi] = \int d^2 z e^{-|z|^2} \psi^\dagger(z) e^{2p\alpha(\bar{z})} \ddagger \xi(\partial_{\bar{z}}, \bar{z}) \ddagger \; e^{-2p\alpha(\bar{z})} \psi(\bar{z}) \tag{2.2}$$

where $\alpha(\bar{z}) = \int d^2 z' e^{-|z'|^2} \ln(\bar{z} - \bar{z}') \psi^\dagger(z') \psi(\bar{z}')$ and $e^{\alpha(\bar{z})}$ is the quasihole operator since

$$e^{\alpha(\bar{z})} |\Psi\rangle = \int d^2 z_1 ... d^2 z_N e^{-\sum_i |z_i|^2} F(\bar{z}_1, ..., \bar{z}_N) \prod_i (\bar{z} - \bar{z}_i) | z_1 \cdots z_N \rangle \tag{2.3}$$

with $|\Psi\rangle = \int d^2 z_1 ... d^2 z_N e^{-\sum_i |z_i|^2} F(\bar{z}_1, ..., \bar{z}_N) | z_1 \cdots z_N\rangle$.

The operators W_{2p} satisfy a strong W_∞ algebra[6,7]

$$[W_{2p}[\xi_1], W_{2p}[\xi_2]] = W_{2p}[\{\!\{ \xi_1, \xi_2 \}\!\}] \tag{2.4}$$

They further play the role of a spectrum generating algebra in the space of Laughlin states and in particular the ground state is the highest weight state

$$(W_{2p})_{lk} |\Psi_{\nu=1/m}\rangle_0 = 0 \qquad\qquad \text{if} \quad l > k \tag{2.5}$$

This provides an algebraic statement of incompressibility for the Laughlin ground states. The operators W_{2p} form a one-parameter family of W_∞ representations.

These ideas can be now extended[7] to include other incompressible states corresponding to filling fractions $\nu \neq 1/m$. In general all these states can be characterized as the highest weight states of different realizations of a W_∞ algebra. I shall briefly present three such cases and explicitly write down the second quantized expressions of the corresponding W_∞ generators 1) $\nu = 1 - 1/m$ states 2) multilayer systems and 3) Jain states.

2.1. $\nu = 1 - 1/m$ states

Using the idea of particle-hole conjugation[17], we can write the $\nu = 1 - \frac{1}{m}$ ground state, in the thermodynamic limit and up to normalization factors, as

$$|\Psi_{\nu=1-1/m}\rangle_0 \sim \int d^2 z_1 ... d^2 z_M e^{-\sum |z_i|^2} \prod_{i<j} (z_i - z_j)^m \psi(\bar{z}_1) ... \psi(\bar{z}_M) |\Psi_{\nu=1}\rangle_0 \tag{2.6}$$

Introducing the operator $\beta(z) = \int d^2z' e^{-|z'|^2} \ln(z-z')\psi(\bar{z}')\psi^\dagger(z')$, we find that

$$\tilde{W}_{2p}[\xi] = \int d^2z e^{-|z|^2} \psi(\bar{z}) e^{2p\beta(z)} \ddagger \xi(\partial_z, z) \ddagger\ e^{-2p\beta(z)} \psi^\dagger(z) \tag{2.7}$$

satisfy a W_∞ algebra and the state in Eq. (2.6) satisfies a highest weight condition. The operator $\tilde{W}_{2p}$ is essentially the charge-conjugated version of W_{2p}.

2.2. Multilayer states

In order to obtain more general filling fractions, one may consider systems where several distinct species of electrons are involved. Wavefunctions of the form

$$\Psi^K(\vec{x}_i^I) = \prod_{I=1}^{r} \prod_{i<j} (\bar{z}_i^I - \bar{z}_j^I)^{K_{II}} \prod_{I<J} \prod_{i,j} (\bar{z}_i^I - \bar{z}_j^J)^{K_{IJ}} e^{-1/2 \sum_{iI} |z_i|^2} \tag{2.8}$$

have been suggested[18] as candidates for describing incompressible Hall states for an r-layer system. K_{II} are odd integers so that the wavefunctions are antisymmetric under exchange of identical fermions. The corresponding filling fraction is given by $\nu = \sum_{I,J}(K^{-1})_{IJ}$ where K is an $r \times r$ symmetric matrix. In order to identify the W_∞ algebra structure associated with Eq. (2.8) we introduce r independent lowest Landau level fermion operators $\psi^I(\bar{z},t)$ and the corresponding quasihole operators $e^{\alpha^I(\bar{z})}$. The W_∞ generators are now[7]

$$W_{\tilde{K}}^I[\xi] = \int d^2z e^{-|z|^2} \psi^{\dagger I}(z) e^{\sum_J \tilde{K}_{IJ}\alpha^J(\bar{z})} \ddagger \xi(\partial_{\bar{z}}, \bar{z}) \ddagger\ e^{-\sum_j \tilde{K}_{IJ}\alpha^J(\bar{z})} \psi^I(\bar{z}) \tag{2.9}$$

where $I = 1, ..., r$ and $\tilde{K} = K - \mathbb{I}$ ($\mathbb{I}$ is the identity matrix). We can show that the operators $W_{\tilde{K}}^I$ give rise to r commuting copies of W_∞ algebras and the corresponding ground states satisfy highest weight conditions.

The W_∞ generators of Eq. (2.9) can also be written as

$$W_{\tilde{K}}^I[\xi] = \int d^2z e^{-|z|^2} \psi^{\dagger I}(z) \ddagger \xi(\partial_{\bar{z}} - \sum_J \tilde{K}_{IJ} \int d^2z' \frac{\rho^J(z',\bar{z}')}{(\bar{z}-\bar{z}')}, \bar{z}) \ddagger\ \psi^I(\bar{z}) \tag{2.10}$$

The term $\sum_J \tilde{K}_{IJ} \int d^2z' \frac{\rho^J(z',\bar{z}')}{(\bar{z}-\bar{z}')}$ plays the role of a gauge potential and it is similar to the one induced by the Chern-Simons interaction. Analogous first quantized expressions were also derived in ref.15.

2.3. Jain states

One of the theories proposed to explain the observed fractions for the FQHE is the one suggested by Jain[14]. In this approach the FQHE wavefunction is constructed by attaching an even number of magnetic fluxes to electrons filling an integer number of Landau levels. The incompressibility of the IQHE is thus carried over to the FQHE wavefunctions. As far as the hierarchical filling fractions $\nu = \frac{n}{2pn+1}$ are concerned, the essential idea of this scenario is that the corresponding incompressible FQHE ground state wavefunction Ψ_ν^0, is related to the IQHE wavefunction Ψ_n^0 as

$$\Psi_\nu^0 = \prod_{i<j} (\bar{z}_i - \bar{z}_j)^{2p} \Psi_n^0 \tag{2.11}$$

This relation is a straightforward generalization of Eq. (2.1). Using previous ideas we find that the corresponding W_∞ generators are of the form[7]

$$W^I_{2p} = \int d^2z e^{-|z|^2} \psi^{\dagger I}(z,\bar{z}) e^{2p\sum_J \alpha^J(\bar{z})} \ddagger\xi(\partial_{\bar{z}}, \bar{z}-\partial_z)\ddagger\ e^{-2p\sum_J \alpha^J(\bar{z})} \psi^I(z,\bar{z}) \tag{2.12}$$

where $I = 0, ..., n-1$, give rise to n commuting copies of W_∞ algebras and the corresponding Jain ground state for $\nu = \frac{n}{2pn+1}$ satisfies the highest weight condition.

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